arXiv:1210.7119v1 [math.CO] 26 Oct 2012

RELATING EDELMAN-GREENE INSERTION TO THE LITTLE MAP

ZACHARY HAMAKER AND BENJAMIN YOUNG

Abstract. We show that two algorithms for reduced words in the symmetric group, Edelman-Greene insertion and the Little bijection, are in principle the same map. Unifying the two approaches allows us to prove new properties about each map, and to resolve several conjectures made by Lam and by Little.

1. Introduction

1.1. **Preliminaries.** In this paper, we clarify the relationship between two algorithmic bijections, due respectively to Edelman-Greene [1] and to Little [5], both of which deal with reduced decompositions in the symmetric group, S_n. It is well known that S_n can be viewed as a Coxeter group with the presentation

$$S_n = \langle w_1, w_2, \dots, w_{n-1} \mid w_i^2 = 1,\ w_iw_j = w_jw_i \text{ for } |i-j| \geq 2,\ w_iw_{i+1}w_i = w_{i+1}w_iw_{i+1}\rangle$$

where w_i can be viewed as the transposition $(i\ i{+}1)$. Let $\sigma = \sigma_1\sigma_2\dots\sigma_n \in S_n$. A *reduced decomposition* or *reduced expression* of σ is a minimal-length sequence $w_{a_1}, w_{a_2}, \dots, w_{a_m}$ such that $\sigma = w_{a_1}w_{a_2}\dots w_{a_m}$. The word $w = a_1a_2\dots a_m$ is called a *reduced word* of σ. It is convenient to refer to a reduced decomposition by its corresponding reduced word and we will conflate the two often. The set of all reduced decompositions of σ is denoted $\mathrm{Red}(\sigma)$. An *inversion* in σ is a pair (i,j) with $i<j$ and $\sigma_i > \sigma_j$. Let $l(\sigma)$ be the number of inversions in σ. Since each transposition w_i either introduces or removes an inversion, for $w = a_1\dots a_m$ a reduced word of σ, we see $m = l(\sigma)$.

The enumerative theory of reduced decompositions were first studied in [7], where using algebraic techniques it is shown for the reverse permutation $\sigma = n\dots 21$ that

$$|\mathrm{Red}(\sigma)| = \frac{\binom{n}{2}!}{(2n-3)(2n-5)^2\dots 5^{n-2}3^{n-2}}. \tag{1}$$

This is the same as the number of standard Young tableaux with the staircase shape $\lambda = (n-1, n-2, \dots, 1)$. In addition, Stanley conjectured for arbitrary $\sigma \in S_n$ that $|\mathrm{Red}(\sigma)|$ can be expressed as the number of standard Young tableaux of various shapes (possibly with multiplicity). This conjecture was resolved in [1] using a generalization of the Robinson-Schensted insertion algorithm, usually called *Edelman-Greene insertion*. Edelman-Greene insertion maps a reduced word w to the pair of Young tableaux $(P(w), Q(w))$ where the entries of $P(w)$ are row-and-column strict and $Q(w)$ is a standard Young tableau. The same map also provides a bijective proof of (1), as there is only one possibility for $P(w)$.

Algebraic techniques developed in [4] can be used to compute the exact multiplicity of each shape for given σ. A bijective realization of Lascoux and Schützenberger's techniques in this setting is demonstrated in [5]. Permutations with precisely one descent are referred to as *Grassmannian*. There is a simple bijection between reduced words of a Grassmannian permutation σ and standard Young tableaux of a shape determined by σ. The Little map works by applying a sequence of modifications referred to as *Little bumps* to the reduced word w until the modified word's corresponding permutation is Grassmannian so that it can be mapped to a standard Young tableau denoted $\mathrm{LS}(w)$.

1.2. **Results.** Since the Little map's introduction, there has been speculation on its relationship to Edelman-Greene insertion. In the appendix of [2], written by Little, Conjecture 4.3.2 asserts that $\mathrm{LS}(w) = Q(w)$ when the maps are restricted to reduced words which realize the reverse permutation. Similar comments are made in [5]. We show the connection is much stronger than previously suspected: this equality is true for every permutation.

Date: October 29, 2012.

Theorem 1.1. *Let w be a reduced word. Then*

$$Q(w) = LS(w).$$

The proof is based on an argument from canonical form. We define the *column word*, a new reading word of $P(w)$ that plays nice with both Edelman-Greene insertion and Little bumps. We then show the statement's truth is invariant under Coxeter-Knuth moves, transformations that span the space of reduced words with identical $P(w)$.

Given Theorem 1.1, one might suspect the structure of the two maps is intimately related. Specifically, Conjecture 2.5 of [3] proposes that Little bumps relate to Edelman-Greene insertion in a way that is analogous to the role dual Knuth transformations play for the Robinson-Schensted-Knuth algorithm.

Let v and w be reduced words. We say v and w *communicate* if there exists a sequence of Little bumps changing v to w. This is an equivalence relation as Little bumps are invertible.

Theorem 1.2 (Lam's Conjecture)**.** *Let v and w be two reduced words. Then v and w communicate if and only if $Q(v) = Q(w)$.*

1.3. **Structure of the paper.** In the second section, we review those parts of [1, 5] which we need: we define Edelman-Greene insertion and the Little map, as well as generalized Little bumps. Additionally, we state some properties of these maps that are important to our work. The third section defines Coxeter-Knuth transformations and studies their interaction with Little bumps and action on $Q(w)$. We conclude in the fourth section by proving our main results and resolving several conjectures of Little.

2. Two Maps

2.1. **Edelman-Greene insertion.** In order to define Edelman-Greene insertion, we must first define a rule for inserting a number into a tableau. Let $n \in \mathbb{N}$ and T be a tableau with rows $R_1, R_2, \ldots, R_k$ where $R_i = r_1^i \leq r_2^i \leq \cdots \leq r_{l_i}^i$. We define the insertion rule for Edelman-Greene insertion, following [1].

(1) If $n \geq r_{l_1}^1$ or if R_i is empty, adjoin k to the end of R_i.
(2) If $n < r_{l_1}^1$, let j be the smallest number such that $n < r_j^1$.
   (a) If $r_j^1 = n + 1$ and $r_{j-1}^1 = n$, insert $n + 1$ into $T' = R_2, \ldots, R_k$ and leave R_1 unchanged.
   (b) Otherwise, replace r_j^1 with n and insert it into $T' = R_2, \ldots, R_k$.

Aside from 2(a), this is the RSK insertion rule. For $w = w_1 \ldots w_m$ a word (not necessarily reduced), we define $\mathrm{EG}(w) = (P(w), Q(w))$ via the following sequence of tableaux (see Figure 1 for an example). We obtain $P_1(w)$ by inserting a_m into the empty tableau. Then $P_j(w)$ is obtained by inserting a_{m-j+1} into $P_{j-1}(w)$. Note we are inserting the entries of w from right to left. At each step, one additional box is added. In $Q(w)$, the entry of each box records the time of the step in which it was added. From this, we can conclude that $Q(w)$ is a standard Young tableau. Note the fourth insertion in Figure 1 follows 2(a). For w is a reduced word of some σ, it is shown that the entries of $P(w)$ are strictly increasing across rows and down columns in [1]. Additionally, we can recover σ from $P(w)$ with no additional information.

2.2. **Grassmannian permutations.** Recall a permutation σ is Grassmannian if it has exactly one descent. We can then write

$$\sigma = a_1 a_2 \ldots a_k b_1 b_2 \ldots b_{n-k}$$

where $\{a_i\}_{i=1}^k$ and $\{b_j\}_{j=1}^{n-k}$ are increasing sequences with $a_{n-k} > b_1$. A word w is *Grassmannian* if it is the reduced word of a Grassmannian permutation. From the Grassmannian word $w = w_1 \ldots w_m$ we construct a tableau $\mathrm{Tab}(w)$ as follows. Index the columns of $\mathrm{Tab}(w)$ by $b_1, \ldots, b_{n-k}$ and the rows by $a_k, a_{k-1}, \ldots, a_1$. Since all inversions in σ feature an a_i and a b_j, each w_l in w represents the swap between an a_i and a b_j. For w_l, we enter $m+1-l$ in the column indexed by a_i and b_j. If a_i swaps with b_j, we see it must later swap with each smaller b. This shows entries are increasing across rows. Likewise, if b_j swaps with a_i, it must later swap with each larger a so entries increase down columns. From this, we can conclude that $\mathrm{Tab}(w)$ is a standard Young tableau whose shape is determined by σ. For a given Grassmannian permutation σ, this map is a bijection as the process is easily reversed. Multiple Grassmannian permutations may correspond

FIGURE 1. Edelman-Greene insertion for $w = 4, 2, 1, 2, 3, 2, 4$

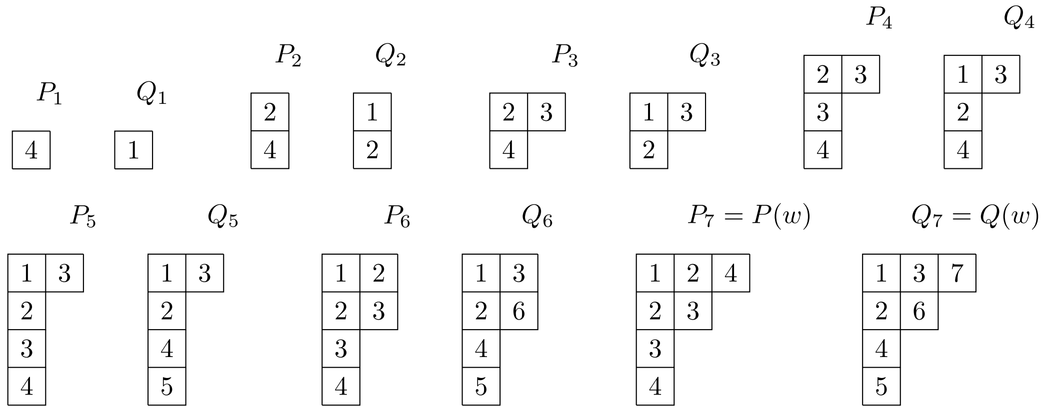
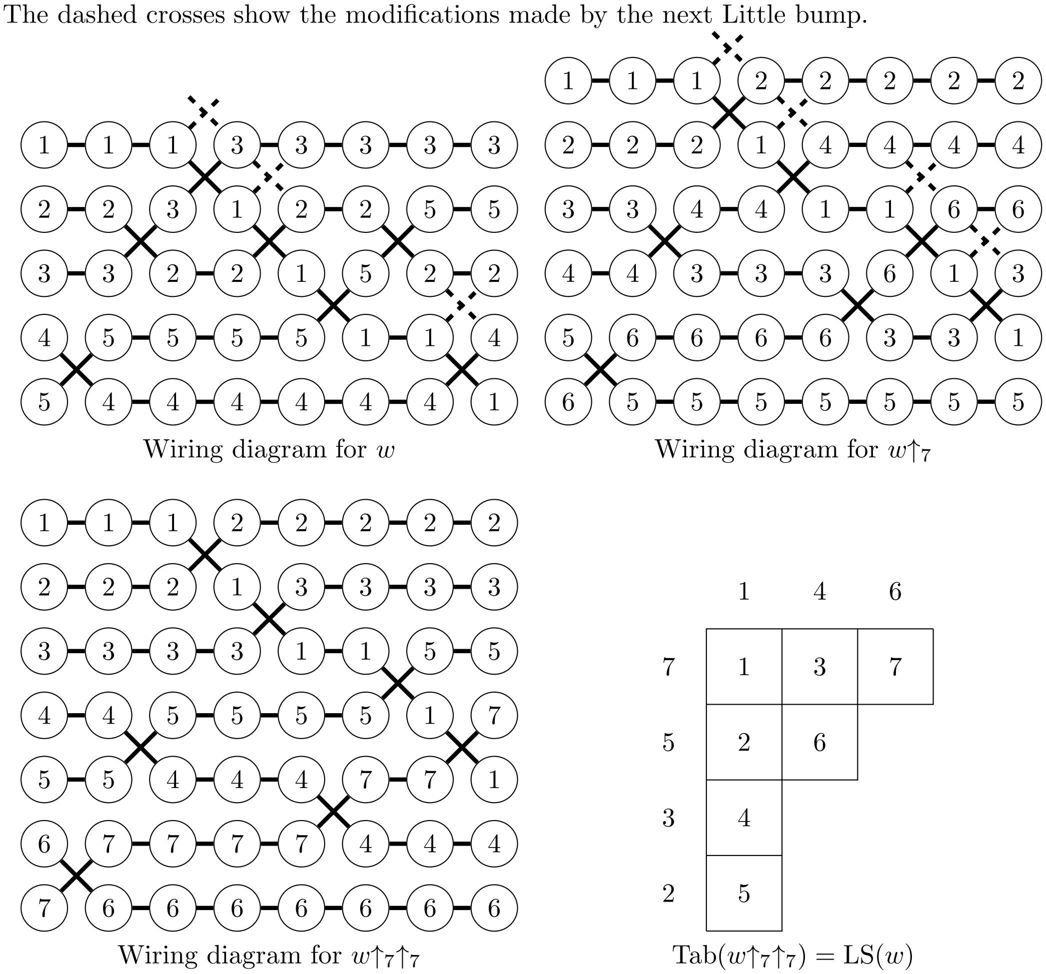


FIGURE 2. The Little map for the reduced decomposition $w_4w_2w_1w_2w_3w_2w_4$ of $\sigma = 35241$. The dashed crosses show the modifications made by the next Little bump.



to the same shape. However, they will only differ by some fixed points at the beginning and end of the permutation.

2.3. **Little bumps and the Little map.** We now describe the method in [5] for transforming an arbitrary reduced word into the reduced word of a Grassmannian permutation. Let $w = w_1 \dots w_m$ be a reduced word and $w^{(i)} = w_1 \dots w_{i-1} w_{i+1} \dots w_m$. We construct

$$w^{(i-)} = \begin{cases} w_1 \dots w_{i-1}(w_i - 1)w_{i+1} \dots w_m & \text{if } w_i > 1 \\ (w_1+1) \dots (w_{i-1}+1) w_i (w_{i+1}+1) \dots (w_m+1) & \text{if } w_i = 1 \end{cases}$$

by decrementing w_i by one or incrementing each other entry if $w_i = 1$.

Let w be a reduced word so that $w^{(i)}$ is also reduced. Note $w^{(i-)}$ may not be reduced, as $w_i - 1$ may swap the same values as some w_j with $j \neq i$. However, this is the only way $w^{(i)-}$ can fail to be reduced as $w^{(i)}$ is reduced and we have added one additional swap. Removing w_j from $w^{(i-)}$, we obtain a new reduced word $w^{(i-)(j)}$. Repeating this process of decrementation, we can construct $w^{(i-)(j-)}$ and so on until we are left with a reduced word $v = v_1 \dots v_m$. We refer to this process as a *Little bump* beginning at position i and say $v = w{\uparrow_i}$, where i is the initial index the bump was started at. To see that this process terminates, we refer to the following lemma.

Lemma 2.1 (Lemma 5, [5])**.** *Let w be a reduced word such that $w^{(i)}$ is reduced. Let $i_1, i_2, \dots$ be the sequence of indices decremented in $w{\uparrow_i}$. Then the entries of $i_1, i_2, \dots$ are unique.*

Since w is finite, we see the process terminates so that $w{\uparrow_i}$ is well-defined. We highlight a property of Little bumps observed in [5], that they preserve the descent structure of w.

Corollary 2.2. *Let $w = w_1 \dots w_m$ and $v = v_1 \dots v_m$ be a reduced words and $\uparrow$ be a Little bump such that $v = w\uparrow$. Then $v_i > v_{i+1}$ if and only if $w_i > w_{i+1}$ for all i.*

Proof. Let $w_i > w_{i+1}$. As each w_i is decremented at most once, we see $v_i \geq v_{i+1}$, but $v_i \neq v_{i+1}$. Thus $v_i > v_{i+1}$. By the same reasoning, if $w_i < w_{i+1}$, we see $v_i < v_{i+1}$. $\square$

Let w be a reduced word of $\sigma \in S_n$. We define the Little map $\mathrm{LS}(w)$.

(1) If w is a Grassmannian word, then $\mathrm{LS}(w) = \mathrm{Tab}(w)$
(2) If w is not a Grassmannian word, identify the swap location i of the last inversion (lexicographically) in σ and output $\mathrm{LS}(w{\uparrow_i})$.

It is a corollary of work in [4] and [5] that LS terminates. We then see that $w \mapsto \mathrm{LS}(w)$ where $\mathrm{LS}(w)$ is a standard Young tableau. An example can be seen in Figure 2, where the word w is represented by its *wiring diagram*: an arrangement of horizontal, parallel wires spaced one unit apart, labelled 1 through n on the left-hand side, in which the letter in the word w are represented by crossings of wires.

3. The action of Coxeter-Knuth moves

3.1. **Basics of Coxeter-Knuth moves.** First introduced in [1], Coxeter-Knuth moves are perhaps the most important tool for studying Edelman-Greene insertion. They are modifications of the second and third Coxeter relations. Let $a < b < c$ and x be integers. The three *Coxeter-Knuth moves* are the modifications

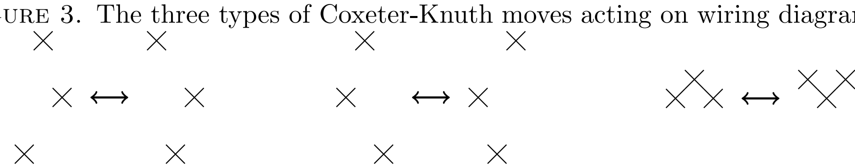
(1) $acb \leftrightarrow cab$
(2) $bac \leftrightarrow bca$
(3) $x(x+1)x \leftrightarrow (x+1)x(x+1)$

applied to three consecutive entries of a reduced word. Let $ww_1w_2 \dots w_m$ be a reduced word of σ and α_i denote a Coxeter-Knuth move on the entries $w_{i-1}w_iw_{i+1}$. Since $a < b < c$, if α_i is of type one or two we have $w\alpha_i$ a reduced word of σ as well by the second Coxeter relation. If α_i is of type three then $w\alpha_i$ is a reduced word of σ by the third Coxeter relation. We say two reduced words v and w are *Coxeter-Knuth equivalent* if there exists a sequence $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}$ of Coxeter-Knuth moves such that

$$v = w\alpha_{i_1} \dots \alpha_{i_k}.$$

Note that two Coxeter-Knuth equivalent reduced words must be correspond to reduced decompositions of the same permutation. We can see their action on wiring diagrams in Figure 3.

FIGURE 3. The three types of Coxeter-Knuth moves acting on wiring diagrams.



(a) Type 1 (b) Type 2 (c) Type 3

Coxeter-Knuth moves play a role in the study of Edelman-Greene insertion analogous to that of Knuth moves in the study of RSK insertion.

Theorem 3.1 (Theorem 6.24 in [1]). *Let v and w be a reduced words. Then $P(v) = P(w)$ if and only if v and w are Coxeter-Knuth equivalent.*

3.2. **The action of Coxeter-Knuth moves on $Q(w)$.** In order to understand the relationships of Coxeter-Knuth moves and Little bumps, we must first understand in greater detail how Coxeter-Knuth moves relate to Edelman-Greene insertion. From Theorem 3.1, we understand how Coxeter-Knuth moves relate to $P(w)$. We must also understand their action on $Q(w)$. For T a standard Young tableau with n entries, let $Tt_{i,j}$ be the Young tableau obtained by swapping the entries labeled $n-i$ and $n-j$.

Lemma 3.2. *Let $w = w_1 \dots w_m$ be a reduced word and α be a Coxeter-Knuth move on $w_{i-1}w_iw_{i+1}$. If α is a Coxeter-Knuth move of type one or three, then*

$$Q(w\alpha) = Q(w)t_{i-1,i}.$$

If α is a Coxeter-Knuth move of type two, then α_i acts on $Q(w)$ as above or

$$Q(w\alpha) = Q(w)t_{i,i+1}.$$

Proof. For $w = w_1 \dots w_m$ a reduced word we see $w|_{i-1} = w_{i-1}w_i \dots w_m$ is also a reduced word. Let α_i be a Coxeter-Knuth move on $w_{i-1}w_iw_{i+1}$. Then

$$P(w|_{i-1}) = P(w|_{i-1}\alpha_i) = P(w\alpha_i \,|_{i-1})$$

since they differ by a Coxeter-Knuth move. Since $w_1 \dots w_{i-2}$ remain unmodified, they insert the same in both cases. Additionally, we see $P(w\,|_{i+2}) = P(w\alpha_i\,|_{i+2})$, so all changes in $Q(w)$ must occur at the entries labeled $i-1, i$ and $i+1$. The remainder of this argument is adapted from the proof for Theorem 6.24 in [1].

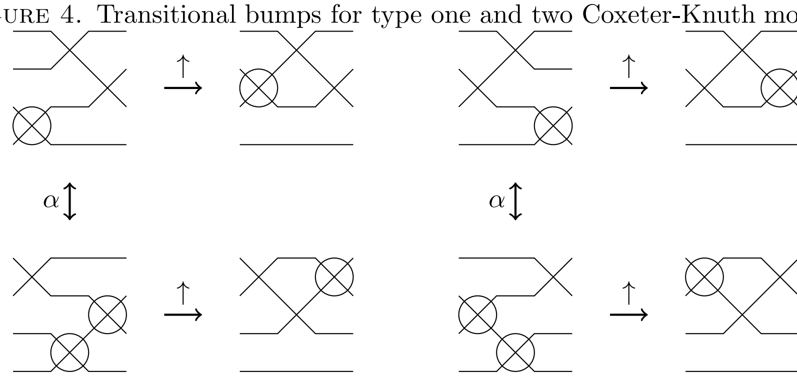
(1) Let α be a Coxeter-Knuth move of type one. Then w_{i+1} inserts into the same spot in $P(w\,|_{i+2})$ for both w and $w\alpha_i$. Since $Q(w) \neq Q(w\alpha_i)$, we see $Q(w\alpha_i) = Q(w)t_{i-1,i}$.

(2) Let α be a Coxeter-Knuth move of type three. This case is treated first as the case with moves of type two relies on it. We compare the insertion of $x(x+1)x$ and $(x+1)x(x+1)$ into the same row of $P(w|_{i+2})$. Assume both x and $x+1$ bump an entry of the row. Let p denote the entry bumped by x, ϵ_1 be the entry preceding p and ϵ_2 be the entry following p. If $p > x+1$, we see x and $x+1$ are inserted into the same position, so $Q(w\alpha) = Q(w)t_{i-1,i}$. Let $p = x+1$. Since $w|_{i+2}$ is reduced, $\epsilon_2 = x+2$. There are two possibilities. We examine the case where $\epsilon_1 < x$. Upon inserting $x(x+1)x$ into the row, we see the first x bumps $x+1$, $x+1$ bumps $x+2$ and the second x bumps the $x+1$ just inserted, so that $(x+1)(x+2)(x+1)$ is inserted into the next row. Upon inserting $(x+1)x(x+1)$ into the row, we see the first $x+1$ produces a special bump of $x+2$, the x bumps $x+1$ and the second $x+1$ bumps the $x+2$ remaining after the special bump, so that $(x+2)(x+1)(x+2)$ is inserted into the next row. The case where $\epsilon_1 = x$ is left to the reader, and has an identical outcome.

If one of the three inserted letters does not bump an entry of the row, we see the largest entry k of the row must be less than $x+1$. As $P(w\alpha|_{i+1})$ is row and column strict, we see $k < x$, so x or $x+1$ would both insert at the end of the row. Thus $Q(w\alpha) = Q(w)t_{i-1,i}$.

(3) Let α be a Coxeter-Knuth move of type two. We compare the insertion of $w_{i-1}w_iw_{i+1}$ and $w_{i-1}w_iw_{i+1}$ into $P(w|_{i+2})$ with $w_i < w_{i+1}$. In the first case, let w_{i+1} bump p, w_i bump q and w_{i-1} bump r. In the latter, let w_i bump p', w_{i+1} bump q' and w_{i-1} bump r', so that we compare the insertion of pqr and $p'q'r'$ into the next row. If $p = p'$, we see $Q(w\alpha) = Q(w)t_{i-1,i}$ as the first entry is inserted into the same spot. Assume $p' < p$. The reader can verify that pqr and $p'q'r'$ differ by a

Figure 4. Transitional bumps for type one and two Coxeter-Knuth moves.



Coxeter-Knuth move of type two unless $p = q + 1$, so that $c = p$. In this case, pqr and $p'q'r'$ differ by a Coxeter-Knuth move of type three.

If some letter does not bump an entry of the row, there are two possibilities. Let k be the largest entry of the row. If $k < w_i$, then w_i and w_{i+1} are inserted into the same position, so $Q(w\alpha) = Q(w)t_{i-1,i}$. If $w_i < k < w_{i+1}$, then w_{i+1} inserts on the end of the row and w_i bumps the same entry of the row regardless of the order of insertion, so $P(w|_i) = P(w\alpha|_i)$. Therefore, $Q(w\alpha) = Q(w)t_{i,i+1}$.

□

3.3. **Coxeter-Knuth moves and Little bumps.** We now set out to show that Coxeter-Knuth moves commute with Little bumps. This requires two results. The first is that the order we perform a Coxeter-Knuth move α and a Little bump $\uparrow$ does not affect the resulting reduced word.

Lemma 3.3. *Let $w = w_1 \dots w_m$ be a reduced word, α a Coxeter-Knuth move on $w_{i-1}w_iw_{i+1}$, and $\uparrow_{j,k}$ be a Little bump begun at the swap between the j and kth trajectories. Then*

$$(w\alpha){\uparrow_{j,k}} = (w{\uparrow_{j,k}})\alpha.$$

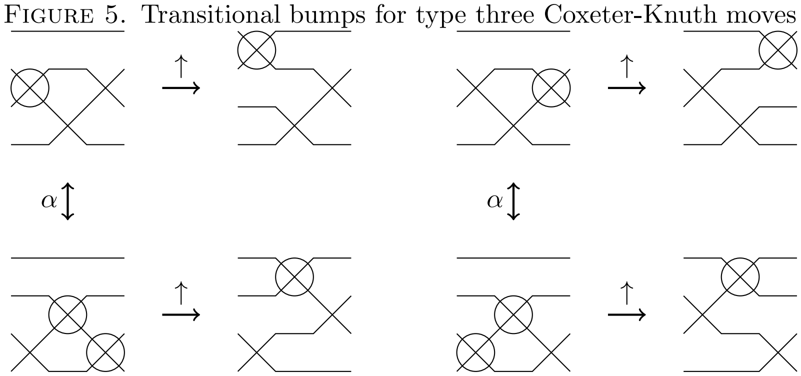
Proof. Let $v = w{\uparrow_{j,k}}$ and $v' = (w\alpha){\uparrow_{j,k}}$. Recall from Lemma 2.1 and Corollary 2.2 that $w_j - v_j \in \{0, 1\}$ and v has the same descent structure of w.

(1) Let α be a Coxeter-Knuth move of the first type, i.e. $w_{i-1}w_iw_{i+1} \mapsto w_iw_{i-1}w_{i+1}$ with w_{i+1} strictly between w_{i-1} and w_i. Since a Little bump decrements an entry of w by at most one, one can check that if w_{i+1} differs from w_i or w_{i-1} by more than one, there is a Coxeter-Knuth move of type one on $v_{i-1}v_iv_{i+1}$. In the event that they differ by exactly one and the smallest entry is decremented, we see in Figure 4 that after the bump they differ by a Coxeter-Knuth move of the third type.
(2) Let α be a Coxeter-Knuth move of the second type, i.e. $w_{i-1}w_iw_{i+1} \mapsto w_{i-1}w_{i+1}w_i$ with w_{i-1} strictly between w_{i+1} and w_i. Since a Little bump decrements an entry of w by at most one, one can check that if w_{i-1} differs from w_i or w_{i+1} by more than one, there is a Coxeter-Knuth move of type two on $v_{i-1}v_iv_{i+1}$. In the event that they differ by exactly one and the smallest entry is bumped, we see in Figure 5 that after the bump they differ by a Coxeter-Knuth move of the third type.
(3) Let α be a Coxeter-Knuth move of the third type. Note the middle entry cannot be bumped unless all three entries are bumped. In the event fewer entries (but not zero) are bumped, we see in Figure 5 that there will be a Coxeter-Knuth move of the first or second type remaining.

We next show that the rest of the Little bump proceeds in the same manner once the crossings involved in the Coxeter-Knuth move have been bumped. To see this, we need only observe that the last bumped swap is between the same two trajectories. This can be verified readily by examining Figures 4 and 5.

The preceding argument assumes that the bumping path does not return to the crossings involved in the Coxeter-Knuth move. It is possible that the bumping path passes through the crossings involved in the Coxeter-Knuth path twice (but no more than that, by Lemma 2.1). However, the same argument applies,

FIGURE 5. Transitional bumps for type three Coxeter-Knuth moves



showing that all three crossings are bumped regardless of whether the Coxeter-Knuth move is performed before or after the bump.

□

We now show that the action of a Coxeter-Knuth move on $Q(w)$ remains the same after applying a Little bump. Combined with Lemma 3.3, this shows that the order in which Coxeter-Knuth moves and Little bumps are performed on a reduced word w does not effect either the resulting reduced word or the resulting recording tableau.

Lemma 3.4. *Letw be a reduced word, α be a Coxeter-Knuth move and $\uparrow$ a Little bump. Then $Q(w\alpha) = Q(w)t_{i,i+1}$ if and only if $Q(w\uparrow\alpha) = Q(w\uparrow)t_{i,i+1}$.*

Proof. By Lemma 3.2, we see α must exchange $w_{i-1}w_iw_{i+1}$ or $w_iw_{i+1}w_{i+2}$. We show the result in the case where α is a Coxeter-Knuth move on $w_{i-1}w_iw_{i+1}$, so that α is a Coxeter-Knuth move of type two. The other outcome then follows.

Let $w' = w\alpha$. Then $w|_i = w_iw_{i+1}w_{i+2}\dots w_n$ and $w'|_i = w_{i+1}w_iw_{i+2}\dots w_n$ are the parts of w and w' respectively to the right of w_{i-1}. Applying Edelman-Greene insertion to $w|_i$ and $w'|_i$, we see $P(w|_i) = P(w'|_i)$ and $Q(w|_i) = Q(w'|_i)t_{i,i+1}$. Therefore, there exists a sequence of Coxeter-Knuth moves $\alpha_1\dots\alpha_m$ such that $w|_i = w'|_i\alpha_1\dots\alpha_m$. We then see

$$Q(w\uparrow|_i) = Q((w'\alpha_1\dots\alpha_m)\uparrow|_i) = Q((w'\uparrow)\alpha_1\dots\alpha_m|_i)$$

by Lemma 3.3. Therefore $w\uparrow|_i$ and $w'\uparrow|_i$ differ solely at their first two positions and are Coxeter-Knuth equivalent, so we see$Q(w\uparrow|_i)$ and $Q(w'\uparrow|_i)$ have the same shape with $Q(w\uparrow|_i) = Q(w'\uparrow|_i)t_{i,i+1}$. Thus $Q(w\uparrow)$ and $Q(w'\uparrow)$ vary in the same way as $Q(w)$ and $Q(w')$.

Since the inverse of a Little bump is a Little bump of the upside down word, where all Coxeter-Knuth move types are preserved, the converse holds as well. Therefore $Q(w\alpha) = Q(w)t_{i,i+1}$ if and only if $Q(w\uparrow\alpha) = Q(w\uparrow)t_{i,i+1}$.

□

4. Proof of Results

4.1. **The Grassmannian case.** Before proving Theorem 1.1, we need to establish the base case where w is a Grassmannian word. In order to do so, we must understand which entries are exchanging places with each swap. For $w = w_1\dots w_m$ a reduced word, we define $\sigma_i = s_{w_1}s_{w_2}\dots s_{w_i}$ where σ_0 is the identity permutation. The kth *trajectory* of w is the sequence $\{\sigma_i(k)\}_{i=0}^m$. For w a Grassmannian word of $\sigma = a_1a_2\dots a_kb_1b_2\dots b_{n-k}$, observe that the jth column of $\mathrm{Tab}(w)$ lists the times for all swaps featuring b_j. Since all such swaps increase the value of b_j, we can reconstruct its trajectory from the number and location of these swaps. Similarly, we can reconstruct the trajectory of each a_i from the $k+1-i$th row of $\mathrm{Tab}(w)$. We will find it convenient to identify the kth trajectory of a Grassmannian word with the indices $\{i_1, i_2, \dots, i_{t_k}\} \subset [n]$ of the swaps featuring k. Since insertion takes place from right to left, we label the entries such that $i_1 > i_2 > \cdots > i_{t_k}$.

Lemma 4.1. *Let $w = w_1 \dots w_m$ be a reduced decomposition of a Grassmannian permutation σ. Then $Tab(w) = Q(w)$.*

Proof. Let $\sigma = a_1 a_2 \dots a_{n-k} b_1 b_2 \dots b_k$ be a Grassmannian permutation with sole descent $a_{n-k} b_1$ and $w = w_1 \dots w_m$ a reduced decomposition of σ. Note the trajectories of the b_j's are non-intersecting as no two swap with each other.

We now show that when applying Edelman-Greene insertion to w, if w_k is in the trajectory of b_j, then w_k will be inserted into the jth column of $P_{n+1-k}(w)$ and each entry bumped during this insertion will in turn insert into the jth column. From this, we can conclude that $\mathrm{Tab}(w) = Q(w)$.

If b_1 has the only non-trivial trajectory amongst the b_j, then $Q(w) = \mathrm{Tab}(w)$ trivially: there is only one column in $\mathrm{Tab}(w)$. Assume there are multiple b_j with non-trivial trajectories. Let $\{i_1, i_2, \dots, i_{t_2}\}$ be the trajectory of b_2. Note $w_{i_k} = w_{i_{k+1}} + 1$. Then b_1 has trajectory $\{l_1, \dots, l_{t_1}\}$ with $t_1 \geq t_2$ and $l_k > i_k$, *i.e.* the kth from last swap featuring b_1 comes later than the kth from last featuring b_2 and so on. Inserting from right to left, we see that upon inserting any $w_{i_{t_2}}$, we will have already inserted $w_{l_{t_1}}$. Therefore, $w_{i_{t_2}}$ will be inserted into the second column as any previously inserted entry will be from the trajectory of b_1, and thus insert into the first column. When $w_{i_{t_2 - 1}}$ is inserted, it too will insert into the second column as $w_{l_{t_1 - 1}}$ will have been inserted into the first column. For identical reasons as before, $w_{i_{t_2}}$ will remain in the second column upon being bumped. We then see inductively that, unimpeded by other swaps, the trajectory of b_2 will insert one after another into the second column. The same argument applies to b_3 and so on. Thus $\mathrm{Tab}(w) = Q(w)$. $\square$

4.2. **The column reading word.** The only ingredient missing from our argument is a canonical form that is invariant under Little bumps.

Definition 4.2. For T a Young tableau with columns $C^1, C^2 \dots, C^m$ where $C^i = c^i_1, c^i_2, \dots, c^i_k$ with c^i_j being the (j, i)th entry of T. We define the *column reading word* of T to be the word $\tau(T) = C^m C^{m-1} \dots C^1$. Note if T is row and column strict then $P(\tau(T)) = T$ and each column of $Q(\tau(T))$ has consecutive entries. For w a reduced word, we define $\tau(w)$ to be $\tau(P(w))$. By the previous observation, w and $\tau(w)$ are Coxeter-Knuth equivalent.

One can think of the column reading word as closely related to the bottom-up reading word. Since insertion takes place from right to left, the column reading word is in some sense its transpose.

Lemma 4.3. *Let w be a reduced word and $\uparrow$ a Little bump on w. Then*

$$Q(\tau(w)) = Q(\tau(w)\uparrow).$$

Proof. Let w be a reduced word, $\tau(w) = C^m C^{m-1} \dots C^1$ and $\tau(w)\uparrow = D^m D^{m-1} \dots D^1$ (note D^k is not *a priori* a column of $P(\tau(w)\uparrow)$). Since $\tau(w)$ and $\tau(w)\uparrow$ have the same descent structure, we see C^1 and D^1 insert identically. As each entry of $\tau(w)\uparrow$ is decremented at most once and $P(\tau(w))$ is row and column strict, we see

$$d^k_i \leq c^k_i \leq d^k_i + 1 \leq d^{k+1}_i,$$

so d^{k+1}_i will not bump any d^k_j with $j \leq i$. Therefore, any entry of D^k will stay in the kth column of $P(\tau(w)\uparrow)$ for all k, that is the entries of the kth column of $P(\tau(w)\uparrow)$ are D^k. Thus $\tau(w)\uparrow$ is a column reading word with identical column sizes, so $Q(\tau(w)) = Q(\tau(w)\uparrow)$.

$\square$

4.3. **Proof of Theorem 1.1 and its corollaries.** Combining Lemma 4.3 with Lemmas 3.3 and 3.4, we can conclude the following:

Theorem 4.4. *Let w be a reduced word and $\uparrow$ be a Little bump on w. Then*

$$Q(w) = Q(w\uparrow).$$

Proof. Let w be a reduced word. There exists a sequence $\alpha_1, \alpha_2, \dots, \alpha_k$ of Coxeter-Knuth moves such that $w = \tau(w)\alpha_1 \dots \alpha_k$. As $Q(\tau(w)) = Q(\tau(w)\uparrow)$ by Lemma 4.3, we compute

$$Q(w) = Q(\tau(w)\alpha_1 \dots \alpha_k) = Q((\tau(w)\uparrow)\alpha_1 \dots \alpha_k) \tag{2}$$

$$= Q((\tau(w)\alpha_1 \dots \alpha_k)\uparrow) = Q(w\uparrow) \tag{3}$$

where the third equality follows by Lemmas 3.3 and 3.4. □

Proof of Theorem 1.1. Let w be a reduced word and $\uparrow_1, \ldots, \uparrow_k$ be the sequence of canonical Little bumps. By Theorem 4.4 and Lemma 4.1, we see

$$Q(w) = Q(w\uparrow_1 \ldots \uparrow_k) = \mathrm{Tab}(w\uparrow_1 \ldots \uparrow_k) = \mathrm{LS}(w).$$

□

We now demonstrate several corollaries, including Lam's Conjecture. The first is Conjecture 11 from [6], which first appeared as Conjecture 4.3.3 in the appendix of [2].

Corollary 4.5. *Let w be a reduced word and let $\uparrow_1, \uparrow_2, \ldots, \uparrow_m$ be any sequence of Little bumps such that*

$$v = w\uparrow_1 \ldots \uparrow_m$$

is a Grassmannian word. Then $Tab(v) = LS(w)$.

This follows from Theorem 4.4. We can extend this result further. Let λ be a partition with w a Grassmannian word of shape λ. The permutation σ associated to w can be characterized by the number of initial fixed points and terminal fixed points. A Grassmannian permutation is *minimal* if it has no initial or terminal fixed points. Note the minimal Grassmannian permutation of a given shape is unique. Recall two reduced words *communicate* if there exists a sequence of Little bumps and inverse Little bumps changing one to the other.

Proof of Theorem 1.2. Let v and w be reduced words. Suppose first that v and w communicate. Then by Theorem 4.4, we have that $Q(v) = Q(w)$.

Conversely, suppose that $Q(v) = Q(w)$. By applying the Little map, w can be changed to the Grassmannian word w' and v to the Grassmannian word v' by a sequence of Little bumps. Since $Q(w) = Q(w')$ and $Q(v) = Q(v')$, we can conclude that v and w communicate if Grassmannian permutations of the same shape communicate. To do this, we demonstrate a sequence of Little bumps that adds a fixed point at the end of an arbitrary Grassmannian permutation, and another sequence that converts a fixed point at the beginning into one at the end. By converting any fixed points at the beginning into ones at the end, then removing those at the end via inverse bumps, we get the minimal Grassmannian permutation of that shape. Therefore, any Grassmannian permutation communicates with the minimal permutation of that shape. From this, we can conclude any two Grassmannian permutations with the same shape communicate.

We now construct our sequence of Little bumps. Let $\sigma = a_1 \ldots a_k b_1 \ldots b_{n-k}$ be a Grassmannian permutation with $a_{n-k}b_1$ its sole descent. Start a Little bump at the last swap featuring each b_j, beginning b_1, so that the first bump begins between b_1 and a_k. We will show this sequence of bumps decrements every entry in each trajectory exactly once. This is equivalent to decrementing each entry of w. If σ has initial fixed points, this will remove one of them, leaving a fixed point at the end. If σ has no initial fixed point, this will leave w the same but add a fixed point to the end of σ.

We now verify that our sequence works as described. First, we must verify that the swap locations at which we begin a Little bump are valid choices, that is that removing that swap from w leaves a reduced word. To see this, note that the first such swap chosen is the swap between a_k and b_1, the last swap in w. This bump will decrement every entry in the trajectory of b_1. After the first Little bump, the second swap chosen is the last in the trajectory of b_2. Since the trajectories of all b_j with $j > 2$ are unaffected by the initial Little bump, this is the last swap for both b_2 and a_k, so removing it leaves a reduced word. This bump will decrement every entry in the trajectory of b_2. Note because we have already decremented the swaps in the trajectory of b_1 and these trajectories were initially disjoint, they will remain disjoint after the second Little bump. Applying this line of reasoning inductively, we see that each Little bump in the sequence is a valid Little bump which decrements every entry of each trajectory. We have now shown v and w communicate if $Q(v) = Q(w)$.

□

Additionally, we show how to embed Robinson-Schensted insertion and RSK in the Little map. In doing so, we recover the main results of [6] in a much simplified form. This embedding was first predicted as Conjecture 4.3.1 in the appendix of [2]. For w a word, let $\vec{w}$ be the reverse of w.

Theorem 4.6. *Let $\sigma = \sigma_1 \dots \sigma_n \in S_n$, so that $w(\sigma) = (2\sigma_n - 1)\dots(2\sigma_1 - 1)$ is a reduced word, and let $RS(\sigma) = (P'(\sigma), Q'(\sigma))$ be the output of Robinson-Schensted insertion applied to σ. Upon applying the transformation $k \mapsto k - 1/2$ to the entries of $LS(w)$, we obtain $Q'(\sigma)$. We can obtain $P'(\sigma)$ by applying the same transformation to $LS(w(\sigma^{-1})$.*

Proof. Since $\mathrm{LS}(w) = Q(w)$ and there are no special bumps, Edelman-Greene insertion will perform the same insertion process on w as Robinson-Schensted insertion performs on σ. Therefore, upon applying the transformation $k \mapsto k - 1/2$, we see $\mathrm{LS}(w(\sigma)) = Q(w(\sigma)) = Q'(\sigma)$. Since $\mathrm{RS}(\sigma^{-1}) = (Q'(\sigma), P'(\sigma))$ (see *e.g.* [8]), we can obtain $P'(\sigma)$ by applying the same transformation to $\mathrm{LS}(w(\sigma^{-1}))$.

$\square$

We can embed RSK in Robinson-Schensted insertion (see Section 7 of [6] for a description of this process), so Theorem 4.6 recovers an embedding of RSK into the Little map as well.

References

[1] P. Edelman and C. Greene. Balanced tableaux. *Advances in Mathematics*, 63(1):42–99, 1987.

[2] Adriano Garsia. *The Saga of Reduced Factorizations of Elements of the Symmetric Group.* Labaratoire de combanatoire et d'informatique mathématique, 2002.

[3] T. Lam. Stanley symmetric functions and peterson algebras. *Arxiv preprint arXiv:1007.2871*, 2010.

[4] A. Lascoux and M.P. Schützenberger. Schubert polynomials and the littlewood-richardson rule. *letters in mathematical physics*, 10(2):111–124, 1985.

[5] D.P. Little. Combinatorial aspects of the lascoux-schützenberger tree. *Advances in Mathematics*, 174(2):236–253, 2003.

[6] D.P. Little. Factorization of the robinson–schensted–knuth correspondence. *Journal of Combinatorial Theory, Series A*, 110(1):147–168, 2005.

[7] R.P. Stanley. On the number of reduced decompositions of elements of coxeter groups. *Eur. J. Comb.*, 5:359–372, 1984.

[8] R.P. Stanley. *Enumerative combinatorics*, volume 2. Cambridge Univ Pr, 2001.